Lecture

Ch. 7 P. 360
(7.1) (7.2) (7.3).

Eigen values & Eigen vectors :

$A_{2\times2}$ & $A_{3\times2}$

$\Longrightarrow (A - \lambda I) X = 0.$

$|A - \lambda I| = 0 \Longrightarrow$ Charactereristic equati.

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

so we get $\lambda = \lambda_1,$
$\lambda = \lambda_1$

Example :-

Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}$$

Then find A^6.

Sol.

$(A - \lambda I)X = 0$.

$$\begin{bmatrix} -3-\lambda & 4 \\ -5 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$|A - \lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & 4 \\ -5 & 6-\lambda \end{vmatrix} = 0$$

We get the characteristic equation

$(3-\lambda)(6-\lambda)+20=0$

$-18-3\lambda+\lambda^2+20=0$

$\lambda^2-3\lambda+2=0$

$(\lambda-1)(\lambda-2)=0$

$\lambda=1 \qquad \lambda=2 \Longrightarrow$ eigen values.

at $\lambda=1$.

$\begin{bmatrix} -4 & 4 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be same lines

$-4x_1+4x_2=0$

$4x_1=4x_2 \Longrightarrow \dfrac{x_1}{x_2}=\dfrac{1}{1} \Rightarrow P_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

at $\lambda=2 \rightarrow \begin{bmatrix} -5 & 4 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$

$-5x_1 + 4x_2 = 0$

$5x_1 = 4x_2$

$\frac{x_1}{x_2} = \frac{4}{5}$ $\quad P_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$ Eigen vectors

$\lambda = 1, 2.$

$D = P^{-1} AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

diagonal Matrix.

$P^{-1} = \frac{1}{1} \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

when diagonalize the matrix we must

make multiplication

$D = P^{-1} A P \qquad \times (P)$

$PD = \underbrace{P P^{-1}}_{I} A P$

$PD = AP \qquad \times (P^{-1})$

$PDP^{-1} = A \underbrace{P P^{-1}}_{I}$

$\boxed{A = PDP^{-1}}$

$A^2 = A \times A$

$A^2 = PD\underbrace{P^{-1} P}_{I} DP^{-1}$

$PD^2P^{-1} = A^2$ - similar $A^3 = PD^3P^{-1}$

General $A^n = PD^nP^{-1}$

General $A^n = P D^n P^{-1}$

incaseof 3×3.

$$A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} P^{-1}.$$

in case of 2×2.

$$A^n = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}$$

in the example.

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$$

$A^6 = P D^6 P^{-1}$

$$= \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (1)^6 & 0 \\ 0 & (2)^6 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -64 & 64 \end{bmatrix} \begin{bmatrix} 5-256 & -4+256 \\ 5-320 & -4+320 \end{bmatrix}$$

$$= \begin{bmatrix} -251 & 252 \\ -315 & 316 \end{bmatrix} \equiv A \times A \times A \times A \times A \times A$$

Example:

Prove that the matrix A satisfied the characteristic equation and then find A^{-1} to the previous example.

$\therefore$ Char. eq $= \lambda^2 - 3\lambda + 2 = 0$

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$A^2 - 3A + 2I = 0$$

$A \times A$

$$\begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} - 3\begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} +$$

$$2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -11 & 12 \\ -15 & 16 \end{bmatrix} - \begin{bmatrix} -9 & 12 \\ -15 & 18 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[A^2 - 3A + 2I = 0) \quad A^{-1}$$

$$A^{-1}AA - 3A^{-1}A + 2A^{-1}I = 0$$

$$A - 3I + 2A^{-1} = 0 \implies A^{-1} = -\tfrac{1}{2}(A - 3I)$$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = P^{-1}AP = 1/2 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 8 & -8 & 0 \\ -8 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example:

Find the Eigen value & Eigen vector then find A^4

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol.

$(A - \lambda I)X = 0.$

$$\begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

let $|A - \lambda I| = 0.$

Characteristic equation.

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0.$$

$$A^{-1} = -\frac{1}{2}(A - 3I) \quad \text{or} \quad \left[\frac{1}{2}(3I - A)\right]$$

$$A^{-1} = \frac{1}{2}\left\{3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 4 \\ -3 & 6 \end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -4 \\ 5 & 3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \Longrightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -4 \\ 5 & -3 \end{bmatrix}$$

at $\lambda = 0$.

$P_2 \equiv P_3$

$P_3 \equiv P_2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-x_1 + x_2 + 0\,x_3 = 0$$

let $x_2 = t$, $x_3 = s \Rightarrow x_1 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s$$

P_2 $\qquad$ P_3

$$P_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$P = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$(-1-\lambda)(-1-\lambda)(-\lambda) - 1(-\lambda) = 0$

$-\lambda[(-1-\lambda)^2 - 1]$

$-\lambda(\lambda^2 + 2\lambda \cancel{+1} \cancel{-1}) = 0$

$-\lambda \cdot \lambda(\lambda + 2) = 0$

$\lambda = 0, 0, -2$ Ho التالتة مختلفين

طالما اتنين زي بعض

والتالت مختلف

at $\lambda = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 : x_2 : x_3 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} : -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} : \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$

$\therefore x_1 : x_2 : x_3 = 2 : -2 : 0$

$x_1 : x_2 : x_3 = 1 : -1 : 0$

$\therefore P_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \lambda = -2$